J. DIFFERENTIAL GEOMETRY
45 (1997) 336-348

NILPOTENT GROUPS AND UNIVERSAL COVERINGS OF SMOOTH PROJECTIVE VARIETIES

L. KATZARKOV

1. Introduction

Characterizing the universal coverings of smooth projective varieties is an old and hard question. Central to the subject is a conjecture of Shafarevich according to which the universal cover $\widetilde{X}$ of a smooth projective variety is holomorphically convex, meaning that for every infinite sequence of points without limit points in $\widetilde{X}$ there exists a holomorphic function unbounded on this sequence.

In this paper we try to study the universal covering of a smooth projective variety X whose fundamental group $\pi_1(X)$ admits an infinite image homomorphism

$$\rho : \pi_1(X) \longrightarrow L$$

into a complex linear algebraic group L. We will say that a nonramified Galois covering $X' \to X$ corresponds to a representation $\rho : \pi_1(X) \to L$ if its group of deck transformations is $\mathrm{im}(\rho)$.

Definition 1.1. We call a representation $\rho : \pi_1(X) \to L$ linear, reductive, solvable or nilpotent if the Zariski closure of its image is a linear, reductive, solvable or nilpotent algebraic subgroup in L. We call the corresponding covering linear, reductive, solvable or nilpotent respectively.

The natural homomorphism $\pi_1(X,x) \to \hat{\pi}_{\mathrm{uni}}(X,x)$ to Malcev's pro-unipotent completion will be called the Malcev representation and the corresponding covering the Malcev covering.

Received December 1, 1995, and, in revised form, March 3, 1996. The author was partially supported by A.P. Sloan Dissertational Fellowship

One may ask not only if the universal covering of X is holomorphically convex but also if some special intermediate coverings that correspond to representations $\rho : \pi_1(X) \longrightarrow L$ are holomorphically convex.

In case X is an algebraic surface and $\rho : \pi_1(X) \longrightarrow L$ is a reductive representation this question has been answered in [13]. The author and M. Ramachandran proved there that if $X' \longrightarrow X$ is a Galois covering of a smooth projective surface corresponding to a reductive representation of $\pi_1(X)$ and such that $\mathrm{Deck}(X'/X)$ does not have two ends, then X' is holomorphically convex.

In this paper we study the other extreme. Namely we study if nilpotent coverings $X' \to X$ are holomorphically convex for X smooth projective variety. We prove the following:

Theorem 1.1. *The Malcev covering of any smooth projective X is holomorphically convex.*

As an immediate consequence of this statement we have:

Theorem 1.2. *Let X be a smooth projective variety with a virtually nilpotent fundamental group. Then the Shafarevich conjecture is true for X.*

(Recall that a finitely generated group is nilpotent if its lower central series has finitely many terms. A group is virtually nilpotent if it has a finite index subgroup which is nilpotent.)

The proof of Theorem 1.1 uses the functorial Mixed Hodge Structure (MHS) on $\pi_1(X)$ combined with some new ideas of János Kollár from [15] and [16]. At the end of paper we give a different proof of Theorem 1.2. We also give some examples as well as some suggestion for the case of solvable coverings. Combining Theorem 1.1 and the theorem proved by the author and M. Ramachandran in [13] we get the following:

Corollary 1.1. *Let X be a smooth projective surface and $\rho : \pi_1(X) \to L$ be a linear representation with an infinite image. Then the universal covering $\widetilde{X}$ of X admits nontrivial holomorphic functions.*

2. The Malcev covering

In this section we prove Theorem 1.1 and Theorem 1.2.

We start with some ideas of János Kollár from [15] and [16]. In [15] Kollár observed that the Shafarevich conjecture is equivalent to the following:

1) There exist a normal variety $\mathbf{Sh}(X)$ and a proper map with connected fibers $\mathbf{Sh} : X \longrightarrow \mathbf{Sh}(X)$, which contracts precisely the subvarieties Z in X with the property that $\mathrm{im}[\pi_1(Z') \longrightarrow \pi_1(X)]$ is finite. Here Z' denotes a desingularization of Z.

2) $\mathbf{Sh}(\widetilde{X})$ is a Stein space. Here we denote by $\mathbf{Sh}(\widetilde{X})$ the Grauert-Remmert reduction of $\widetilde{X}$. In our notation $\mathbf{Sh}(X) = \mathbf{Sh}(\widetilde{X})/\pi_1(X)$. The action of $\pi_1(X)$ may have fixed points on $\mathbf{Sh}(\widetilde{X})$ but we can still take a quotient.

One can consider also a relative version of condition 1). Let $H \triangleleft \pi_1(X)$ be a normal subgroup. We will say that a subgroup $R \subset \pi_1(X)$ is almost contained in H if the intersection $R \cap H$ has finite index in R and we will write $R \lesssim H$. We have the following condition.

1. There exist a normal variety $\mathbf{Sh}^H(X)$ and a proper map with connected fibers $\mathbf{Sh}^H : X \longrightarrow \mathbf{Sh}^H(X)$, which contracts exactly the subvarieties Z in X having the property that $\mathrm{im}[\pi_1(Z') \longrightarrow \pi_1(X)] \lesssim H$. Again Z' denotes a desingularization of Z. The relative version of 2) is the following:

2. $\mathbf{Sh}^H(\widetilde{X})$ is a Stein space. Here we denote by $\mathbf{Sh}^H(\widetilde{X})$ the Cartan-Remmert reduction of $\mathbf{Sh}(\widetilde{X})$. In our notation
$$\mathbf{Sh}^H(X) = \mathbf{Sh}^H(\widetilde{X})/(\pi_1(X)/H).$$

This was also independently observed by F. Campana in [4].

Our approach is that if there is a natural candidate for $\mathbf{Sh}(X)$ it is enough to check condition 1) only for Z - an algebraic curve. This certainly is the case when $\pi_1(X)$ is a nilpotent group. In the simplest case when $\pi_1(X)$ is virtually abelian, one uses the Albanese variety $\mathrm{Alb}(X)$ for $\mathbf{Sh}(X)$.

It is clear that for a smooth projective variety X with $\pi_1(X)$ an infinite nilpotent group the Albanese map:
$$\mathrm{Alb} : X \longrightarrow \mathrm{Alb}(X)$$
has nontrivial image. In other words $\dim_{\mathbb{C}}(\mathrm{im}(\mathrm{Alb})) > 0$.

Moreover if we denote by S the Stein factorization of the Albanese map, then this is a natural candidate for $\mathbf{Sh}(X)$ in case $\pi_1(X)$ is a nilpotent group. Observe that the map
$$X \longrightarrow S$$

contracts all subvarieties Z with the property that $\mathrm{im}[H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})]$ is trivial.

Now using that $\pi_1(X)$ is a nilpotent group and the mixed Hodge structures on its Malcev completion we show that for an algebraic curve Z the fact that $\mathrm{im}[H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})]$ is trivial is equivalent to the fact that $\mathrm{im}[\pi_1(Z) \longrightarrow \pi_1(X)]$ is finite for Z an algebraic curve. We finish the proof by reducing the case when Z is of arbitrary dimension to the case when Z is an algebraic curve.

To prove Theorem 1.1 we need to show again that there is natural candidate for $\mathbf{Sh}^H(X)$, where $H = \ker\rho : \pi_1(X) \longrightarrow \hat{\pi}_{\mathrm{uni}}(X,x)$ of the Malcev representation. Again this candidate is S the Stein factorization of the Albanese map. In the next section we give a different proof of Theorem 1.1, which is basically spelling of the proof we have given already in the language of equivariant harmonic maps.

2.1. Mixed Hodge structure considerations

In this subsection we explain why if $\pi_1(X)$ is a nilpotent group the theory of Mixed Hodge Structures on it implies that $\mathrm{im}[H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})]$ is trivial is equivalent to the fact that $\mathrm{im}[\pi_1(Z) \longrightarrow \pi_1(X)]$ is finite for Z an algebraic curve. For some background one can look at [7], [8] or [10].

For the proof of Theorem 1.1 we need to work with X smooth but for completeness in this section we will require only the MHS on $H^1(X)$ is of weights > 0.

Lemma 2.1. *If Z is a compact nodal curve and $f : Z \longrightarrow X$ is a map to a variety such that MHS on $H^1(X)$ is of weights > 0, then the map*

$$f_* : L(Z,x) \longrightarrow L(X,f(x))$$

is trivial if and only if the map

$$f^* : H^1(X,\mathbb{Q}) \longrightarrow H^1(Z,\mathbb{Q})$$

is trivial. Here $L(Z,x)$ and $L(X,f(x))$ are the corresponding Lie algebras of the unipotent completions $\hat{\pi}_{\mathrm{uni}}(Z,x)$ and $\hat{\pi}_{\mathrm{uni}}(X,f(x))$ of the fundamental groups $\pi_1(Z,x)$ and $\pi_1(X,f(x))$ respectively, and x is a point in Z.

Proof. Observe that the map in unipotent completions determines and is determined by a map on the corresponding Lie algebras:

$$L(Z,x) \longrightarrow L(X,f(x)).$$

First let us consider the case where $H_1(Z)$ is pure of weight -1. This is the case when the dual graph of Z is a tree. By a standard strictness argument [7] the weight filtration on $L(Z,x)$ is its lower central series, and the associated graded Lie algebra is generated by $Gr_{-1}L(Z,x) = H_1(Z,\mathbb{Q})$.

Since

$$L(Z,x) \longrightarrow L(X,f(x))$$

is a morphism of MHS, it is non-zero if and only if the map

$$GrL(Z,x) \longrightarrow GrL(X,f(x))$$

on weight graded quotients is a morphism of MHS. Since

$$Gr_{-1}L(X,f(x)) = H_1(X,\mathbb{Q})/W_{-2},$$

and since $H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})$ is trivial, it follows that $L(Z,x) \longrightarrow L(X,f(x))$ is trivial.

To prove the general case, we take a partial normalization

$$Z' \longrightarrow Z$$

with the property that Z' is connected and such that $H^1(Z')$ is a pure MHS of weight 1.

This can be done as follows. Take a maximal tree T in the dual graph of Z and normalize only those double points corresponding to edges not in T. Then $H_1(Z)$ is pure MHS of weight -1. The previous argument implies that

$$L(Z',x) \longrightarrow L(X,f(x))$$

is trivial.

To complete the proof, note that we have an exact sequence

$$1 \longrightarrow N \longrightarrow \pi_1(Z,x) \longrightarrow \pi_1(\Gamma,*) \longrightarrow 1,$$

where Γ denotes the dual graph of Z and N is the normal subgroup of $\pi_1(Z)$ generated by $\pi_1(Z',x)$. After passing to unipotent completions, we obtain an exact sequence

$$0 \longrightarrow (L(Z',x)) \longrightarrow L(Z,x) \longrightarrow L(\Gamma,*) \longrightarrow 0.$$

This is an exact sequence in the category of Malcev Lie algebras with MHS. The ideal $(L(Z',x))$ generated by $L(Z',x)$ is exactly $W_{-1}L(Z,x)$, so the MHS induced on $L(\Gamma,*)$ is pure of weight 0.

It follows that the homomorphism $L(Z,x) \longrightarrow L(X,f(x))$ induces a homomorphism

$$L(\Gamma,*) \longrightarrow L(X,f(x)).$$

This is a morphism of MHS of (0,0) type. It is injective if and only if the map

$$L(\Gamma,*) = GrL(\Gamma,*) \longrightarrow GrL(X,f(x))$$

is also injective. Since $H_1(X)$ has weights < 0 and $L(\Gamma,*)$ has weight zero, it follows that

$$L(\Gamma,*) = GrL(\Gamma,*) \longrightarrow GrL(X,f(x))$$

is zero. This proves the statement in general. Namely, we have that for any nodal curve (singular, reducible) the map

$$f_* : L(Z,x) \longrightarrow L(X,f(x))$$

is trivial if and only if the map

$$f^* : H^1(X,\mathbb{Q}) \longrightarrow H^1(Z,\mathbb{Q})$$

is trivial. q.e.d.

Lemma 2.2. *Let X be a smooth projective variety with a nilpotent fundamental group $\pi_1(X)$. Then for any algebraic curve $Z \subset X$ the fact* $\mathrm{im}[H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})]$ *is trivial is equivalent to the fact that* $\mathrm{im}[\pi_1(Z) \longrightarrow \pi_1(X)]$ *is finite.*

Proof. Since we can always find a partial normalization $\widetilde{Z} \to Z$ with $\widetilde{Z}$-nodal and $\pi_1(\widetilde{Z},\tilde{x}) \to \pi_1(Z,x)$ surjective it follows from the previous lemma that the map

$$f_* : L(Z,x) \longrightarrow L(X,f(x))$$

is the zero map. Furthermore, if $\pi_1(X)$ is a torsion free nilpotent group, then by definition it embeds in $\pi_{un}(X,f(x))$. It is easy to see that torsion elements of a nilpotent group generate a finite group and hence

$$\pi_1(X,f(x)) \longrightarrow \hat{\pi}_{\mathrm{uni}}(X,f(x))$$

is an embedding up to torsion which proves the lemma. q.e.d.

We have actually proved more. Observed that we have not used the fact that the lower central series of $\pi_1(X)$ has finitely many terms. The strictness property of MHS structures allows us to prove:

Lemma 2.3. *Let X be a smooth projective variety and $\rho : \pi_1(X) \longrightarrow L(X, f(x))$ be the Malcev representation of $\pi_1(X)$. Then for any algebraic curve $Z \subset X$ the fact* $\mathrm{im}[H_1(Z,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})]$ *is trivial is equivalent to the fact that* $\mathrm{im}[\pi_1(Z) \longrightarrow \pi_1(X)/H]$ *is finite. Here H is the kernel of the Malcev representation.*

2.2. A reduction to the case of an algebraic curve

In this section we show how to reduce the argument for Z of arbitrary dimension to Z an algebraic curve.

Lemma 2.4. *Let F be a connected subvariety in X. Then we can find a curve $Z \subset F$ such that $\pi_1(Z)$ surjects on $\pi_1(F)$.*

Proof. If F is a smooth variety the above lemma is just the Lefschetz hyperplane section theorem. Let $F = F_1 + \ldots + F_i$ be singular and with many components of different dimension. Denote by n the normalization $n : F' \longrightarrow F$ of F. In every component of F' after additional desingularization we can find finitely many points x_k, y_k such that $n(x_k) = n(y_k)$ and $\pi_1(F'/(x_k = y_k))$ surjects onto $\pi_1(F)$. The way to do that is to take the Whitney stratification of F and put the points x_k, y_k in every stratum. For some singularities this might not be enough. Then instead of identifying the points we connect them with a rational curve, which maps to a singular point. Now following [9, (ii, 1.1)] we take hypersurfaces with large degrees that pass through the points x_k, y_k and intersect every component of F', F'_l in a curve Z_l such that $Z' = \cup Z_l$ and $\pi_1(Z')$ surjects on $\pi_1(F')$. We make $Z = n(Z')$. Observe that Z might be singular and have many components but it will be connected. q.e.d.

Now we are ready to finish the proof of Theorem 1.1. We start with the Stein factorization of the Albanese map for X

$$\mathrm{Alb} : X \longrightarrow S \longrightarrow \mathrm{im}(\mathrm{Alb}) \subset \mathrm{Alb}(X).$$

Denote by S' the fiber product of the universal covering $\widetilde{\mathrm{Alb}(X)}$ of $\mathrm{Alb}(X)$ and S over $\mathrm{Alb}(X)$. By definition S' is mapped finitely to a closed analytic subset in $\widetilde{\mathrm{Alb}(X)}$ and since $\widetilde{\mathrm{Alb}(X)}$ is a Stein manifold S' is a Stein manifold as well. It follows from the definition of the Albanese morphism that the fibers of the map

$$\mathrm{Alb} : X \longrightarrow S$$

are all subvarieties F in X for which the map $H_1(F,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})$ is trivial. We have shown that the fact that $H_1(F,\mathbb{Q}) \longrightarrow H_1(X,\mathbb{Q})$

is trivial implies that $\mathrm{im}[\pi_1(F) \longrightarrow \pi_1(X)/H]$ is finite (Lemmas 2.3 and 2.4). To finish the proof of Theorem 1.1 we need to observe that S satisfies the conditions for being the Shafarevich variety of X, $S = \mathbf{Sh}^H(X)$; namely:

1) There exists a holomorphic map with connected fibers $X \longrightarrow S$, which contracts only the subvarieties Z in X with the property that $\mathrm{im}[\pi_1(Z) \longrightarrow \pi_1(X)/H]$ is finite.

2) $\mathbf{Sh}^H(\widetilde{X}) = S'$ is a Stein space. q.e.d.

To prove Theorem 1.2 we use the same argument as above but H is a finite group. Actually we have shown more. As we pointed out before we have not used the fact that lower central series of $\pi_1(X)$ has finitely many terms. The only thing that is really needed is that $\pi_1(X)$ has a finite torsion. Therefore we get:

Corollary 2.1. *Let X be a smooth projective variety with a finite torsion virtually residually nilpotent fundamental group. Then the Shafarevich conjecture is true for X.*

3. Some examples

In this section we give some examples and geometric applications of our method. We start with the following result that was also proved by Campana in [5].

Corollary 3.1. *Let X be a smooth projective surface and Γ be the image of $\pi_1(X)$ in $L(X, f(x))$. Let as before S be the Stein factorization of the map $X \longrightarrow \mathrm{Alb}(X)$. After taking an etale finite covering $X'' \longrightarrow X$ the homomorphism $\pi_1(X'') \longrightarrow \Gamma$ factors through the map $\pi_1(S) \longrightarrow \Gamma$.*

Proof. According to [16, 4.8] after taking some etale finite covering $X'' \longrightarrow X$, $\pi_1(X'')$ is the same as the fundamental group of $\pi_1(S)$. This follows from the fact that residually nilpotent groups are residually finite. q.e.d.

Nilpotent Kähler groups were constructed by Sommese and Van de Ven [18], and Campana [5] (look also at [6] for a very nice exposition). The construction goes as follows:

Start with a finite morphism from an abelian variety A to $\mathbb{P}^n$. Now take the preimage X in A of generic abelian d-fold in $\mathbb{P}^n$. Then X has as fundamental group a nonsplit central extension of an abelian group by $\mathbb{Z}$. Let us following [18] give more explicit example. We start with a four-dimensional abelian variety A and a finite morphism f to $\mathbb{P}^4$. Take the Mumford-Horrocks abelian surface Z in $\mathbb{P}^4$ and pull it back to A. Let us call the new surface $f^{-1}(Z)$. The following exact sequence was established in [18]

$$\pi_2(A) \oplus \pi_2(Z) \longrightarrow \pi_2(\mathbb{P}^4) \longrightarrow \pi_1(f^{-1}(Z)) \longrightarrow \pi_1(A) \oplus \pi_1(Z) \longrightarrow 0.$$

In our case this sequence reads as:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(f^{-1}(Z)) \longrightarrow \mathbb{Z}^{12} \longrightarrow 0$$

and shows that $f^{-1}(Z)$ has a two steps nilpotent fundamental group.

As a quick geometric application of Theorem 1.2, we get that if X is a smooth projective variety with an infinite virtually nilpotent fundamental group and such that $\operatorname{rank} \operatorname{Pic}(X) = 1$, then for every subvariety Z in X we have that $\operatorname{im}[\pi_1(Z) \longrightarrow \pi_1(X)]$ is infinite. The proof of the above statement is an easy consequence of [16, Chapter 1].

We give now an idea of an alternative proof of Theorem 1.1, which came from conversations with M. Ramachandran. It is based on the use of $\pi_1(X)$ equivariant harmonic maps to the universal coverings to Higher Albanese varieties defined in [12]. Combined with the strictness property for the nonabelian Hodge theory this seems to be a very promising idea (see [14]).

Denote by G_s the complex form of the Malcev completion of $\pi_1(X)/\Gamma^{s+1}$, where Γ^i are the groups from the lower central series for $\pi_1(X)$ and Γ^s is the smallest nontrivial one. Then G_s is a simply connected, complex nilpotent Lie group. The corresponding Lie algebra g_s has MHS coming from the Lie bracket (see [11]). Denote by F^0G_s the closed subgroup in G_s that corresponds to F^0g_s. Since both F^0G_s and G_s are contractible and the group $\pi_1(X)/\Gamma^{s+1}$ is nilpotent, then it follows from [11] that we have a free action of the corresponding to G_s lattice $G_s(\mathbb{Z})$ on G_s/F^0G_s. In case $\pi_1(X)$ is nilpotent up to a finite index subgroup, $G_s(\mathbb{Z})$ is nothing else but $\pi_1(X)$.

Therefore in the same way as in [13] we obtain a $\pi_1(X)$ equivariant proper horizontal holomorphic map (see [12])

$$\widetilde{X} \longrightarrow G_s/F^0G_s.$$

According to [11] G_s/F^0G_s is biholomorphic to $\mathbb{C}^N$. In the same way as in [13] we construct a strictly plurisubharmonic exhaustion function on $\widetilde{X}$. So $\widetilde{X}$ is holomorphically convex.

Remark 3.1. The above argument is weaker than the argument we have used in the first proof, and cannot be generalized to the case of residually nilpotent groups since in this case G_s/F^0G_s will not be a manifold.

Let us at the end say two words about the case where we have an infinite solvabe representation $\rho : \pi_1(X) \to L$. By theorem of Arapura and Nori [1] all Kähler linear solvable groups are virtually nilpotent. So we have the following:

Corollary 3.2. *Let X be a smooth projective variety with a linear solvable fundamental group. Then the universal covering $\widetilde{X}$ is holomorphically convex.*

We cannot prove solvable analog of Theorem 1.1. The maximum we can do is to realize how close the solvable representations come to nilpotent ones. To be able to do so we need to generalize slightly the result of Arapura and Nori by proving

Theorem 3.1. *If Γ is a quotient of a Kähler group $\pi_1(X)$ so that it is a complex linear solvable group, then there are two possibilities - either Γ is deformable to a virtually nilpotent representation of $\pi_1(X)$ or $\pi_1(X)$ surjects onto the fundamental group of a curve of genus bigger than zero.*

First observe that every Zariski dense representation $\rho : \pi_1(X) \to \Gamma$ to a complex linear solvable group can be deformed to a Zariski dense representation $\rho : \pi_1(X) \to S(\overline{\mathbb{Q}})$ to a linear solvable group $S(\overline{\mathbb{Q}})$ (which we denote again by ρ) defined over $\overline{\mathbb{Q}}$ and having an infinite image.

Proof. (The idea of the proof was suggested to me by T. Pantev.) Denote by Γ the image of the solvable representation $\rho : \pi_1(X) \to L$. We need to show that either Γ is virtually nilpotent or there exists a holomorphic map with connected fibers $f : X \to C$ to a smooth curve C of genus ≥ 1. Let us introduce some notation. For a finitely generated group Γ denote by $\Sigma(\Gamma)$ the set of all special characters of Γ; that is,

$$\Sigma(\Gamma) := \left\{\alpha : \Gamma \to \mathbb{C}^\times \,\middle|\, H^1(\Gamma, \mathbb{C}_\alpha) \neq 0\right\},$$

where $\mathbb{C}_\alpha$ is the one-dimensional Γ-module associated to α. Now we recall the following:

Proposition 3.1. (Arapura-Nori [1]) *Let Γ be a finitely generated $\mathbb{Q}$-linear solvable group. Then the following are equivalent:*

1. *Γ is virtually nilpotent.*

2. $\Sigma(\Gamma)$ *consists of finitely many torsion characters.*

Due to this proposition it is enough to show that either $\Sigma(\Gamma)$ consists of finitely many torsion characters or X has a non-trivial map to a curve of genus larger than zero. Now, since $\pi_1(X)$ surjects on Γ it follows that $\Sigma(\Gamma) \subset \Sigma(\pi_1(X))$ and hence it suffices to show that either $\Sigma(\pi_1(X))$ consists of finitely many torsion characters or X has an irrational pencil.

For a smooth projective variety X denote by $M(X)$ the moduli space of homomorphisms from $\pi_1(X)$ to $\mathbb{C}^\times$. The locus of special characters is a jump locus in $M(X)$ and hence is a subscheme in a natural way. It turns out that $\Sigma(\pi_1(X))$ is actually a smoooth subvariety having very special geometric properties which we are going to exploit. Since the subvariety $\Sigma(\pi_1(X)) \subset M(X)$ is completely canonical, one expects it to have an intrinsic description. One way to construct natural subvarieties in $M(X)$ is via pullbacks; namely, given any surjective morphism $\varphi : X \to Y$ we can pullback the moduli space of characters of $\pi_1(Y)$ to get a subvariety $\varphi^* M(Y) \subset M(X)$. According to [17], Lemma 2.1 and Theorem 6.1 every connected component Σ of the subvariety $\Sigma(\pi_1(X)) \subset M(X)$ is of this kind. More specifically for every such Σ there exist a torsion character $\sigma \in \Sigma$ and a connected abelian subvariety $P \subset \mathrm{Alb}(X)$ so that Σ is the translation of $\varphi^* M(\mathrm{Alb}(X)/P) \subset M(X)$ by σ. Here $\varphi : X \to \mathrm{Alb}(X) \to \mathrm{Alb}(X)/P$ is the composition of the Albanese map and the natural quotient morphism. In particular, $\Sigma(\pi_1(X))$ has a positive dimensional component if and only if its intersection with the set of all unitary characters has a positive dimensional component. Now the Hodge decomposition of the cohomology of a unitary local system implies that unless $\Sigma(\pi_1(X))$ consists of finitely many torsion characters the subvariety of all special line bundles in $\mathrm{Pic}^\tau(X)$ has a positive dimensional component. Indeed, for a unitary character α denote by $\mathbb{L}_\alpha$ the corresponding rank-one local system, and by $L_\alpha = \mathbb{L}_\alpha \otimes_{\mathbb{C}} \mathcal{O}_X$ the corresponding holomorphic line bundle. Now by the Hodge theorem we have

$$h^1(\pi_1(X), \mathbb{C}_\alpha) = h^1(X, \mathbb{L}_\alpha) = h^1(X, L_\alpha) + h^0(X, \Omega^1_X \otimes L_\alpha) = 2h^1(X, L_\alpha),$$

i.e., α is a special character iff the line bundle L_α is special.

Furthermore a theorem of Beauville ([3, Proposition 1]) asserts that the subvariety of $\mathrm{Pic}^0(X)$ consisting of special line bundles is a union of a finite set and the subvarieties of the form $f^*\mathrm{Pic}^0(B)$ where $f : X \to B$ is a morphism with connected fibers to a curve B of genus ≥ 1. Thus

X posseses irrational pencils which finishes the proof of Theorem 1.3.

q.e.d.

The above theorem can be seen as the solvable analog of the theorem of Simpson's that $SL(n,\mathbb{Z})$ is not a Kähler group, $n > 2$. This theorem gives a way of constructing new examples of non-Kähler groups. In particular any group Γ with infinite $H^1([\Gamma,\Gamma],\mathbb{Q})$ possessing a solvable linear quotient defined over $\mathbb{Q}$ that is not virtually nilpotent cannot be Kähler. Or this means that any group Γ with no surjective homomorphisms to a nonabelian free group possessing a solvable linear quotient defined over $\mathbb{Q}$ that is not virtually nilpotent cannot be Kähler (see e.g. [2]).

In the case where there exists a holomorphic map with connected fibers $f : X \to C$ to a smooth curve C of genus ≥ 1 we cannot say much about the holomorphic convexity of the corresponding solvable covering of X.

Acknowledgements

I would like to thank D. Abramovich, A. Beilinson, F. Bogomolov, J. Carlson, K. Corlette, R. Donagi, M. Gromov, S. Gersten, M. Larsen, M. Nori, T. Pantev, C. Simpson, D. Toledo and S. Weinberger for the useful conversations and H. Clemens, P. Deligne, R. Hain, J. Kollár and M. Ramachandran for teaching me all ingredients of the technique used in this paper. Special thanks are due to Professor J. Kollár for inviting me to visit University of Utah, where most of this work was done, and also to the referees of the paper, whose constructive criticism has helped me a lot to write a concise, organized paper.

References

[1] D. Arapura & M. Nori, *Solvable fundamental groups of algebraic vatieties and Kähler manifolds,* In preparation.

[2] D. Arapura, *Geometry of cohomology support loci for local systems 1,* Preprint, Purdue University, 1995.

[3] A. Beauville, *Annulation de H^1 et systemes paracanonidues sur les surfaces,* J. Reine Angew. Math. **388** (1988) 149-157.

[4] F. Campana, *Remarques sur le revetement universal des varietes Kähleriennes compactes,* Bull. Soc. Math. France **122** (1994) 255-284.

[5] ______, *Remarques sur le groupes de Kähleriennes nilpotent,* Ann. Sci. École Norm. Sup. **28** (1995) 307-316.

[6] J. Carlson & D. Toledo, *Quadratic presentations and nilpotent Kähler groups,* J. Geom. Anal. **5** (1995) 359-377.

[7] P. Deligne, *Theorie de Hodge II,* Inst. Hautes Études Sci. Publ. Math. **40** (1971) 5-58.

[8] P. Deligne, P. Griffiths, J. Morgan & D. Sullivan, *Real homotopy theory of Kähler manifolds,* Invent. Math. **29** (1975) 245-274.

[9] M. Goresky & R. MacPherson, *Stratified Morse Theory,* Springer, Berlin, 1988.

[10] R. Hain, *The de Rham homotopy theory of complex algebraic variety. I,* K-Theory **1** (1987) 271-324.

[11] ______, *Higher Albanese manifolds,* Hodge theory, Lecture Notes in Math., Vol. 1246, 1987, 84-92.

[12] R. Hain & S. Zucker, *Unipotent variations of mixed Hodge structures,* Invent. Math. **88** (1987) 83-124.

[13] L.Katzarkov & M. Ramachandran, *On the Shafarevich conjecture for surfaces,* Preprint, 1995.

[14] L.Katzarkov, *On the Shafarevich maps,* to appear in Algebraic Geom. (Proc. Sympos. Pure Math., Santa Cruz, Calif. 1995), Amer. Math. Soc.

[15] J. Kollár, *Shafarevich maps and plurigenera of algebraic varieties,* Invent. Math. **113** (1993) 165-215.

[16] ______, *Shafarevich maps and automorphic forms,* To appear in Math. Notes, Princeton University Press.

[17] C. Simpson, *Subspaces of moduli spaces of rank one local systems,* Ann. Sci. École Norm. Sup. **26** (1993) 361-401.

[18] A. J. Sommese & A. Van de Ven, *Homotopy groups of pullbacks of varieties,* Nagoya Math. J. **102** (1986) 79-90.

Math. Sci. Res. Inst., Berkeley